ADVANCED

Stage 6

Branded with your school emblem.

Calculus (Adv) ← Filter by topic.

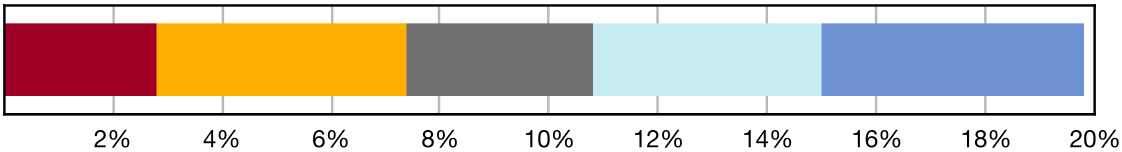
C3 Applications of Calculus (Adv)

Maxima and Minima (Y12)

Teacher: Smarter Maths ← Customised with your name.

Exam Equivalent Time: 60 minutes *(based on allocation of 1.5 minutes per mark)* ← Timed worksheets.

C3 Applications of Calculus



← Analysis.

- The Derivative Function and its Graph
- Curve Sketching
- Tangents
- Maxima and Minima
- Rates of Change

***SmarterMaths analytics based on the average contribution to new syllabus Advanced Maths exams since 2020.**

HISTORICAL CONTRIBUTION

- *C3 Applications of Calculus* is the 800 pound gorilla in the Advanced rainforest, contributing a massive 19.8% to new syllabus exams since introduced in 2020.
- This topic has been split into five sub-topics for analysis purposes: *1-The Derivative Function and its Graph (2.8%), 2-Curve Sketching (4.6%), 3-Tangents (3.4%), 4-Maxima and Minima (4.2%) and 5-Rates of Change (4.8%).*
- This analysis looks at *Maxima and Minima*.

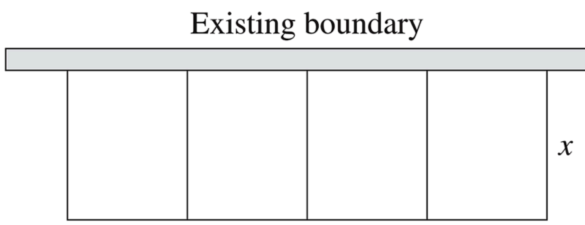
HSC ANALYSIS - What to expect and common pitfalls

- *Maxima and Minima (4.2%)* appears in every exam with significant mark allocations almost guaranteed. Its importance in achieving a band 5-6 result cannot be overstated.
- *Max/Min* problems will typically ask students to prove an equation in the earlier parts. A band 4 level task that presents a good opportunity for all students to score well.
- HSC Markers remind students to work with the given equation of the question, *even if they couldn't successfully complete the proof in the earlier part*.
- *Max/Min* problems have had a number of specific themes. *Area/Volume equations* have dominated, appearing in 4 out of 5 new syllabus exams. Other themes include *distance* (2017 and 2013) and a *cost equation* (2009).
- *Max/Min* was novelly examined within the *Probability Density Function* topic in 2021 and is included in the database under that topic.

Questions

1. Calculus, 2ADV C3 2016 HSC 14c

A farmer wishes to make a rectangular enclosure of area 720 m². She uses an existing straight boundary as one side of the enclosure. She uses wire fencing for the remaining three sides and also to divide the enclosure into four equal rectangular areas of width x m as shown.



i. Show that the total length, l m, of the wire fencing is given by

$l = 5x + \dfrac{720}{x}$. ***(1 mark)***

ii. Find the minimum length of wire fencing required, showing why this is the minimum length. ***(3 marks)***

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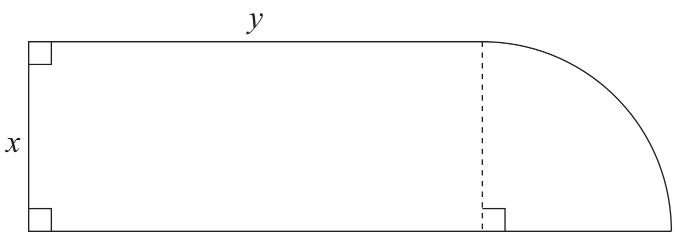
2. Calculus, 2ADV C3 2020 HSC 25

A landscape gardener wants to build a garden in the shape of a rectangle attached to a quarter-circle. Let x and y be the dimensions of the rectangle in metres, as shown in the diagram.



The garden bed is required to have an area of 36 m² and to have a perimeter which is as small as possible. Let P metres be the perimeter of the garden bed.

a. Show that $P = 2x + \dfrac{72}{x}$. ***(3 marks)***

b. Find the smallest possible perimeter of the garden bed, showing why this is the minimum perimeter. ***(4 marks)***

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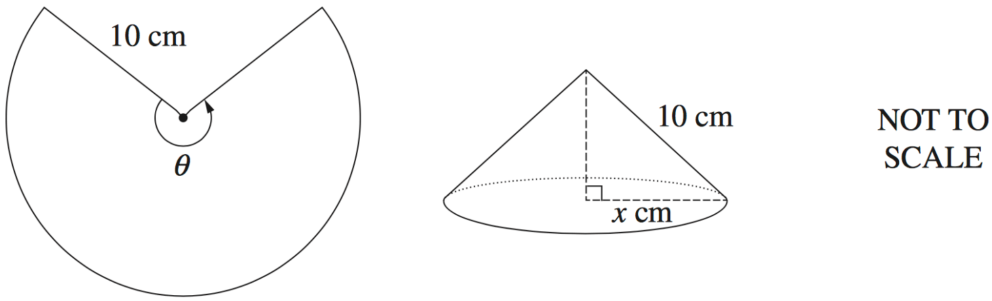
3. Calculus, 2ADV C3 2018 HSC 16a

A sector with radius 10 cm and angle θ is used to form the curved surface of a cone with base radius x cm, as shown in the diagram.



The volume of a cone of radius r and height h is given by $V = \dfrac{1}{3}\pi r^2 h$.

i. Show that the volume, V cm³, of the cone described above is given by

$$V = \frac{1}{3}\pi x^2 \sqrt{100 - x^2}.$$ ***(1 mark)***

ii. Show that $\dfrac{dV}{dx} = \dfrac{\pi x\left(200 - 3x^2\right)}{3\sqrt{100 - x^2}}$. ***(2 marks)***

iii. Find the exact value of θ for which V is a maximum. ***(3 marks)***

Past HSC RAP
data integration.

4. Calculus, 2ADV C3 2019 HSC 15c

Part ii: RAP Data (vs state mean) - Bottom 13%: School result (23%) was 8% below state average (31%)

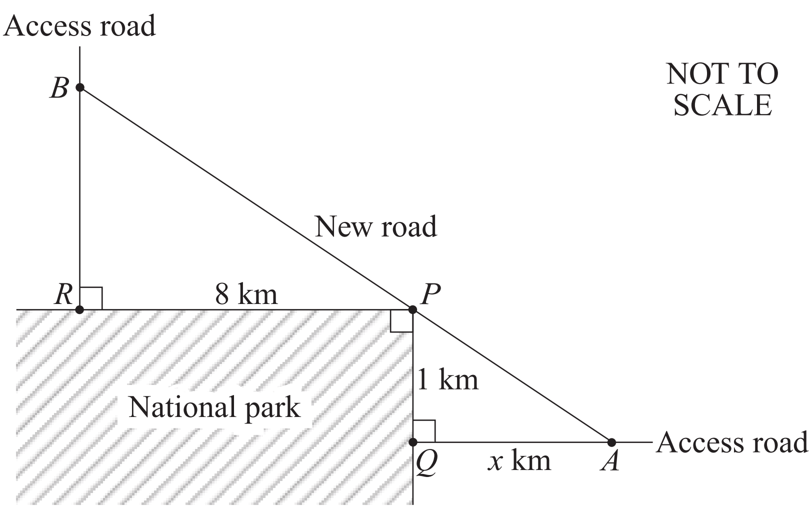
Part ii: RAP Data (raw) - School result was 23%

The entry points, R and Q, to a national park can be reached via two straight access roads. The access roads meet the national park boundaries at right angles. The corner, P, of the national park is 8 km from R and 1 km from Q. The boundaries of the national park form a right angle at P.

A new straight road is to be built joining these roads and passing through P.

Points A and B on the access roads are to be chosen to minimise the distance, D km, from A to B along the new road.

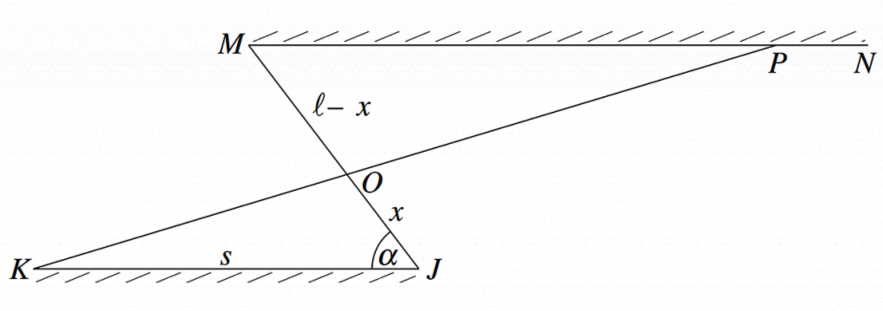
Let the distance QA be x km.



i. Show that $D^2 = (x+8)^2 + \left(\frac{8}{x}+1\right)^2$. ***(3 marks)***

ii. Show that $x = 2$ gives the minimum value of D^2. ***(3 marks)***

5. Calculus, 2ADV C3 2008 HSC 10b



The diagram shows two parallel brick walls KJ and MN joined by a fence from J to M. The wall KJ is s metres long and $\angle KJM = \alpha$. The fence JM is l metres long.

A new fence is to be built from K to a point P somewhere on MN. The new fence KP will cross the original fence JM at O.

Let $OJ = x$ metres, where $0 < x < l$.

i. Show that the total area, A square metres, enclosed by ΔOKJ and ΔOMP is given by

$$A = s\left(x - l + \frac{l^2}{2x}\right)\sin\alpha.$$ ***(3 marks)***

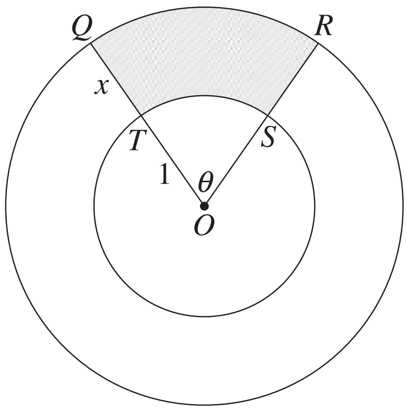
ii. Find the value of x that makes A as small as possible. Justify the fact that this value of x gives the minimum value for A. ***(3 marks)***

iii. Hence, find the length of MP when A is as small as possible. ***(1 mark)***

6. Calculus, 2ADV C3 2024 HSC 31

Two circles have the same centre O. The smaller circle has radius 1 cm, while the larger circle has radius $(1+x)$ cm. The circles enclose a region $QRST$, which is subtended by an angle θ at O, as shaded.

The area of $QRST$ is A cm^2, where A is a constant and $A > 0$.



Let P cm be the perimeter of $QRST$.

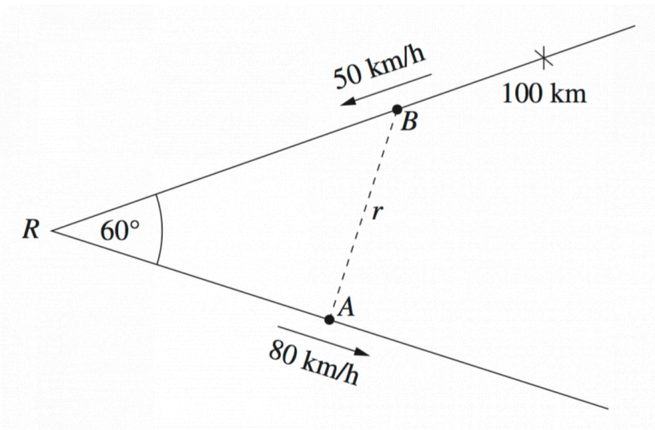
a. By finding expressions for the area and perimeter of $QRST$, show that $P(x) = 2x + \dfrac{2A}{x}$. ***(3 marks)***

b. Show that if the perimeter, $P(x)$, is minimised, then θ must be less than 2. ***(3 marks)***

7. Calculus, 2ADV C3 2013 HSC 14b

Part ii: RAP Data (vs state mean) - Bottom 18%: School result (21%) was 6% below state average (27%)
Part i: RAP Data (raw) - School result was 23%
Part ii: RAP Data (raw) - School result was 21%

Two straight roads meet at R at an angle of 60°. At time $t = 0$ car A leaves R on one road, and car B is 100km from R on the other road. Car A travels away from R at a speed of 80 km/h, and car B travels towards R at a speed of 50 km/h.



The distance between the cars at time t hours is r km.

i. Show that $r^2 = 12\ 900t^2 - 18\ 000t + 10\ 000.$ ***(2 marks)***

ii. Find the minimum distance between the cars. ***(3 marks)***

Worked Solutions

1. Calculus, 2ADV C3 2016 HSC 14c

i. $\text{Area} = xy$

$$720 = xy$$

$$y = \frac{720}{x}$$

Best practice worked solutions.

$$l = 5x + y$$

$$= 5x + \frac{720}{x} \quad \ldots \text{as required}$$

ii. $\dfrac{dl}{dx} = 5 - \dfrac{720}{x^2}$

$$\frac{d^2l}{dx^2} = \frac{1440}{x^3}$$

$$\text{Max/Min when } \frac{dl}{dx} = 0,$$

$$5 = \frac{720}{x^2}$$

$$x^2 = 144$$

$$x = 12, \quad x > 0$$

$$\text{When } \ x = 12, \ \frac{d^2l}{dx^2} > 0$$

$\therefore$ Minimum occurs when $x = 12$

$\therefore$ Minimum fencing

$$= 5 \times 12 + \frac{720}{12}$$

$$= 120 \text{ m}$$

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2. Calculus, 2ADV C3 2020 HSC 25

a. Area $= xy + \dfrac{1}{4}\pi r^2$

$$36 = xy + \frac{1}{4}\pi x^2$$

$$xy = 36 - \frac{\pi x^2}{4}$$

$$y = \frac{36}{x} - \frac{\pi x}{4}$$

♦ *Mean mark part (a) 47%.*

Black diamond icons and "mean marks" highlight difficult past questions.

$$\therefore P = 2x + 2y + \frac{1}{4}(2\pi x)$$

$$= 2x + 2\left(\frac{36}{x} - \frac{\pi x}{4}\right) + \frac{\pi x}{2}$$

$$= 2x + \frac{72}{x} - \frac{\pi x}{2} + \frac{\pi x}{2}$$

$$= 2x + \frac{72}{x}$$

b. $P = 2x + \dfrac{72}{x}$

$$\frac{dP}{dx} = 2 - 72x^{-2}$$

$$\frac{d^2P}{dx^2} = 144x^{-3}$$

Max or min when $\dfrac{dP}{dx} = 0$:

$$2 - \frac{72}{x^2} = 0$$

$$2x^2 = 72$$

$$x^2 = 36$$

$$x = 6, \quad (x > 0)$$

When $x = 6$,

$$\frac{d^2P}{dx^2} = \frac{144}{6^3} = \frac{2}{3} > 0$$

$\Rightarrow$ MIN at $x = 6$

$$\therefore P_{\min} = 2 \times 6 + \frac{72}{6}$$

$$= 24 \text{ m}$$

3. Calculus, 2ADV C3 2018 HSC 16a

i. $V = \frac{1}{3}\pi r^2 h$

Using Pythagoras,

$h = \sqrt{100 - x^2}$

$r = x$

$\therefore$ Volume $= \frac{1}{3}\pi x^2 \sqrt{100 - x^2}$

♦ *Mean mark (ii) 45%.*

ii.

$$V = \frac{1}{3}\pi x^2 \sqrt{100 - x^2}$$

$$\frac{dV}{dh} = \frac{1}{3}\pi \left[2x \cdot \sqrt{100 - x^2} - 2x \cdot \frac{1}{2}\left(100 - x^2\right)^{-\frac{1}{2}} \cdot x^2 \right]$$

$$= \frac{1}{3}\pi \left[\frac{2x\left(100 - x^2\right) - x^3}{\sqrt{100 - x^2}} \right]$$

$$= \frac{1}{3}\pi \left[\frac{200x - 2x^3 - x^3}{\sqrt{100 - x^2}} \right]$$

$$= \frac{\pi x\left(200 - 3x^2\right)}{3\sqrt{100 - x^2}} \quad \text{.. as required}$$

iii. Find x when $\frac{dV}{dx} = 0$

♦♦ *Mean mark (iii) 23%.*

$200 - 3x^2 = 0$

$x = \sqrt{\frac{200}{3}}$

When $x < \sqrt{\frac{200}{3}}$, $\left(200 - 3x^2\right) > 0$

$\Rightarrow \frac{dV}{dx} > 0$

When $x > \sqrt{\frac{200}{3}}$, $\left(200 - 3x^2\right) < 0$

$\Rightarrow \frac{dV}{dx} < 0$

$\therefore$ MAX when $x = \sqrt{\frac{200}{3}}$

Equating the arc length of the section

to the circumference of the cone:

$$
\begin{aligned}
2\pi r \cdot \frac{\theta}{2\pi} &= 2\pi \cdot x \\
10\theta &= 2\pi\sqrt{\frac{200}{3}} \\
\therefore \theta &= \frac{2\pi \cdot 10\sqrt{2}}{10 \cdot \sqrt{3}} \\
&= \frac{2\sqrt{2}\pi}{\sqrt{3}}
\end{aligned}
$$

4. Calculus, 2ADV C3 2019 HSC 15c

i. $\angle BRP = \angle PQA = 90^\circ$ (given)

$\angle BPR = \angle PAQ$ (corresponding)

$\Rightarrow \Delta\ BRP\ |||\ \Delta PQA$ (equiangular)

$$\frac{BR}{RP} = \frac{PQ}{QA} \quad \text{(corresponding sides of similar triangles)}$$

$$\frac{BR}{8} = \frac{1}{x}$$

$$BR = \frac{8}{x}$$

Using Pythagoras,

$$D^2 = (QA + RP)^2 + (BR + PQ)^2$$

$$= (x+8)^2 + \left(\frac{8}{x} + 1\right)^2$$

♦ *Mean mark part (i) 41%.*

ii. $$(D^2)\prime = 2 \cdot (x+8) + 2(-1) \cdot \frac{8}{x^2}\left(\frac{8}{x} + 1\right)$$

$$= 2x + 16 - \frac{16}{x^2}\left(\frac{8}{x} + 1\right)$$

$$= 2x + 16 - \frac{128}{x^3} - \frac{16}{x^2}$$

$$(D^2)\prime\prime = 2 - (-3) \cdot \frac{128}{x^4} - (-2) \cdot \frac{16}{x^3}$$

$$= 2 + \frac{384}{x^4} + \frac{32}{x^3}$$

When $x = 2$,

$$(D^2)\prime = 2 \times 2 + 16 - \frac{128}{8} - \frac{16}{4} = 0$$

♦♦ *Mean mark part (ii) 31%.*

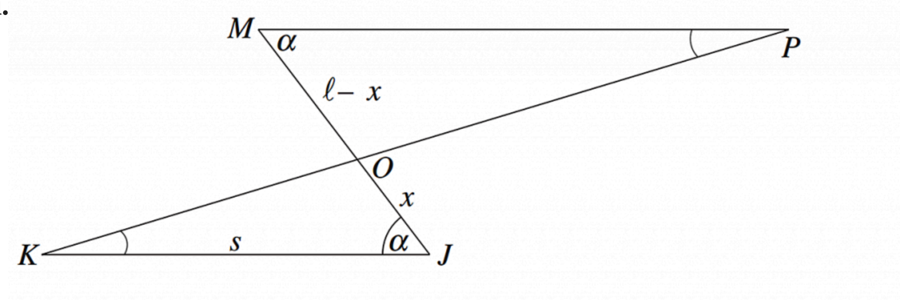
$\Rightarrow$ S.P. at $x = 2$

$$(D^2)\prime\prime = 2 + \frac{384}{16} + \frac{32}{8} > 0$$

$\therefore$ MIN value of D^2 at $x = 2$.

5. Calculus, 2ADV C3 2008 HSC 10b

i.



$A =$ Area $\Delta OJK +$ Area ΔOMP

Using sine rule

Area $\Delta OJK = \dfrac{1}{2}\, xs\sin\alpha$

♦♦♦ Low mean marks highlighted (although exact data not available before 2009).

Area $\Delta OMP \Rightarrow$ Need to find MP

$$\angle OKJ = \angle MPO \quad \text{(alternate angles, } MP \parallel KJ)$$

$$\angle PMO = \angle OJK = \alpha \quad \text{(alternate angles, } MP \parallel KJ)$$

$$\therefore\ \Delta OJK\ |||\ \Delta OMP \quad \text{(equiangular)}$$

$$\Rightarrow \frac{x}{s} = \frac{l-x}{MP} \qquad \text{(corresponding sides of similar triangles)}$$

$$MP = \frac{l-x}{x}\cdot s$$

$$\begin{aligned} \text{Area } \Delta\ OMP &= \frac{1}{2}(l-x)\cdot MP \cdot \sin\alpha \\ &= \frac{1}{2}(l-x)\cdot\frac{(l-x)}{x}\cdot s\ \sin\alpha \end{aligned}$$

$$\begin{aligned} \therefore A &= \frac{1}{2}x\cdot s\sin\alpha + \frac{1}{2}(l-x)\cdot\frac{(l-x)}{x}\cdot s\sin\alpha \\ &= s\sin\alpha\left(\frac{1}{2}x + \frac{1}{2}(l-x)\cdot\frac{(l-x)}{x}\right) \\ &= s\sin\alpha\left(\frac{1}{2}x + \frac{(l-x)^2}{2x}\right) \\ &= s\sin\alpha\left(\frac{1}{2}x + \frac{l^2}{2x} - l + \frac{1}{2}x\right) \\ &= s\left(x - l + \frac{l^2}{2x}\right)\sin\alpha \quad \ldots\text{as required} \end{aligned}$$

ii. Find x such that A is a minimum

$$A = s\left(x - l + \frac{l^2}{2x}\right)\sin\alpha$$

$$\frac{dA}{dx} = s\left(1 - \frac{l^2}{2x^2}\right)\sin\alpha$$

$$\text{MAX/MIN when } \frac{dA}{dx} = 0$$

$$s\left(1 - \frac{l^2}{2x^2}\right)\sin\alpha = 0$$

$$\frac{l^2}{2x^2} = 1$$

$$2x^2 = l^2$$

$$x^2 = \frac{l^2}{2}$$

$$x = \frac{l}{\sqrt{2}}, \quad x > 0$$

$$\frac{d^2A}{dx^2} = s\left(\frac{l^2}{2x^3}\right)\sin\alpha$$

Since $0 < \alpha < 90^\circ \Rightarrow \sin\alpha > 0, \; l > 0$ and $x > 0$

$$\frac{d^2A}{dx^2} > 0 \quad \Rightarrow \text{MIN at } x = \frac{l}{\sqrt{2}}$$

iii. Since $MP = \frac{(l - x)}{x}s$ and MIN when $x = \frac{l}{\sqrt{2}}$

$$MP = \left(\frac{l - \frac{l}{\sqrt{2}}}{\frac{l}{\sqrt{2}}}\right)s \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\left(\sqrt{2}l - l\right)}{l}s$$

$$= \left(\sqrt{2} - 1\right)s \text{ metres}$$

$\therefore \; MP = \left(\sqrt{2} - 1\right)s$ metres when A is a MIN.

MARKER'S COMMENT: Students who could not complete part (i) are reminded that they can still proceed to part (ii) and attempt to differentiate the result given. Note that l and α are constants when differentiating.

Marker's Comments appear when they add value.

6. Calculus, 2ADV C3 2024 HSC 31

a. Area $=$ Sector OQR $-$ Sector OTS

$$A = \frac{\theta}{2\pi} \times \pi(1+x)^2 - \frac{\theta}{2\pi} \times \pi \times 1^2$$

$$A = \frac{\theta}{2}(1+x)^2 - \frac{\theta}{2}$$

$$A = \frac{\theta}{2}(1 + 2x + x^2 - 1)$$

$$A = \frac{\theta}{2}x(x+2)$$

$$\theta = \frac{2A}{x(x+2)}$$

$$\begin{aligned}\text{Perimeter } QRST &= \frac{\theta}{2\pi} \times 2\pi(x+1) + \frac{\theta}{2\pi} \times 2\pi(1) + 2x \\ &= \theta(x+1) + \theta + 2x \\ &= \theta(x+2) + 2x \\ &= \frac{2A}{x(x+2)}(x+2) + 2x \\ &= 2x + \frac{2A}{x}\end{aligned}$$

♦ Mean mark (a) 41%.
COMMENT: Sector/arc calculations used in solution are for those who don't want to remember formulas.

b. $P(x) = 2x + \dfrac{2A}{x}$

$$P'(x) = 2 - \frac{2A}{x^2}$$

$$P''(x) = \frac{4A}{x^3} > 0 \ \ (x, A > 0)$$

$\Rightarrow$ MIN when $P'(x) = 0$:

$$2 - \frac{2A}{x^2} = 0$$

$$2x^2 = 2A$$

$$x^2 = A$$

Using part (a):

$$\theta = \frac{2A}{x(x-2)}$$

$$= \frac{2x^2}{x(x+2)}$$

♦♦♦ Mean mark (b) 12%.

$$= \frac{2x}{x+2}$$

$$= 2 \times \frac{x}{x+2}$$

$$< 2 \ \left(\text{since } \frac{x}{x+2} < 1 \ \text{ for all } x\right)$$

$\therefore$ If $P(x)$ is minimised, $\theta < 2$.

7. Calculus, 2ADV C3 2013 HSC 14b

i. Need to show $r^2 = 12\ 900t^2 - 18\ 000t + 10\ 000$

$RB = 100 - 50t$

♦♦ *Mean mark 26%*

$RA = 80t$

Using the cosine rule

$$\begin{aligned} r^2 &= (RB)^2 + (RA)^2 - 2(RB)(RA)\cos\angle R \\ &= (100-50t)^2 + (80t)^2 - 2(100-50t)(80t)\cos 60 \\ &= 10\ 000 - 10\ 000t + 2500t^2 + 6400t^2 - 8000t + 4000t^2 \\ &= 12\ 900t^2 - 18\ 000t + 10\ 000 \quad \ldots \text{as required} \end{aligned}$$

ii. Max/min when $\dfrac{dr^2}{dt} = 0$

$$\frac{dr^2}{dt} = 25\ 800t - 18\ 000 = 0$$

$$t = \frac{18\ 000}{25\ 800} = \frac{30}{43}$$

When $t = \dfrac{30}{43}$, $\dfrac{d^2(r^2)}{dt^2} = 25\ 800 > 0 \quad \Rightarrow \text{MIN}$

$\therefore$ Minimum distance when $t = \dfrac{30}{43}$ hr

♦♦ *Mean mark 27%*
ALGEBRA TIP: Finding the derivative of r^2 (rather than making r the subject), makes calculations ***much easier***. *ENSURE* you apply the test to confirm a minimum.

Find r when $t = \dfrac{30}{43}$

$$\begin{aligned} r^2 &= 12\ 900\left(\frac{30}{43}\right)^2 - 18\ 000\left(\frac{30}{43}\right) + 10\ 000 \\ &= 3720.9302\ldots \\ \therefore\ r &= \sqrt{3702.9302}\ldots \\ &\approx 60.99942\ldots \\ &\approx 61\ \text{km (nearest km)} \end{aligned}$$

$\therefore$ MIN distance between the cars is 61 km.